SONOMA STATE UNIVERSITY M*A*T*H COLLOQUIUM
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Hard Copy of Transparencies
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OUTLINE
1. What is an Ordinary Differential Equation?
2. What are Growth and Decay Equations?
3. What are Functional Differential Equations?
4. Delay Differential Equations.
6. The Method of Steps for Solving DDE's.
7. Using EXCEL to solve DDE's
8. Pollution Clean up problem
9. Higher Order Equations
10. Other Applications
    The Inverted Pendulum
    Acceleration Delays -- Earthquake Damage Control

DIFFERENTIAL EQUATIONS

Differential equations are used to solve problems such as the following. You know (or think you know) that the "instantaneous rate of change" $x'(t)$ of some quantity $x(t)$ is related to that quantity or some function of that quantity.

$$x'(t) = f(t, x(t)) \qquad (1)$$

EXAMPLE:

If $x(t)$ is the population of some bacteria at time t, and the bacteria grows at a rate $x'(t)$ that is proportional to its own population at the time t, then we write:

$$x'(t) = k\, x(t),\ t \geq 0 \qquad (2)$$

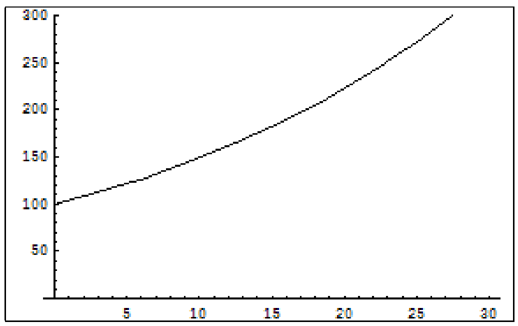
Where k is the constant of proportionality. The general solution to this equation is

$$x(t) = Ce^{kt} \qquad (3)$$

Where C is the population at time $t = 0$.

$x'(t) = k\,x(t) \Rightarrow \frac{x'(t)}{x(t)} = k \Rightarrow ln[x(t)] = kt + ln[C] \Rightarrow x(t) = Ce^{kt}$

For example if $k = .04$ then the population is changing by 4% at time t. The following graph is for an initial value of 100 and over time from 0 to 30



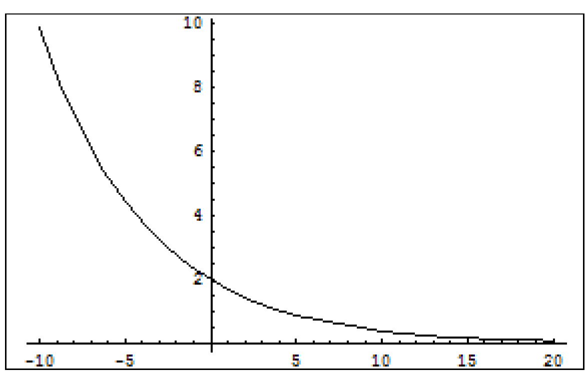
$$x(t) = 100\,e^{.04\,t}$$

RADIO-ACTIVE DECAY

If the quantity being studied is *decaying* instead of *growing* we could get the following equation which says the change is proportional to the amount of material present at time t.

$$x'(t) = -0.16\,x(t) \qquad (4)$$

with initial condition $x(0) = 2$
Solution graph:



Solution to $x'(t) = -0.16\,x(t)$
$x(t) = 2\,e^{-0.16t}$

FUNCTIONAL DIFFERENTIAL EQUATIONS

The differential equation

$$x'(t) = f(t, x(t))$$

is the old stuff, started by the inventors of calculus and differential equations in the 1600's. Cavalieri (1635), Fermat (1636), Decartes (1630), Newton(1687) and Leibniz (1672) , the Bernoulli's (1695-1748), and others.

The new stuff involves equations like

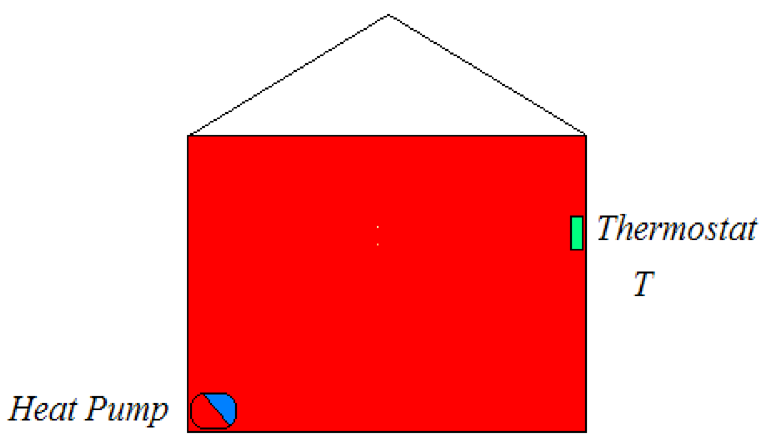
$$x'(t) = f(t, x(u(t)))$$

started by
Volterra (1909), Schmidt (1911), Tychonov (1938), Wright (1948), Krasovkii (1963), El'sgol'ts (1973), and thousands of others since then.

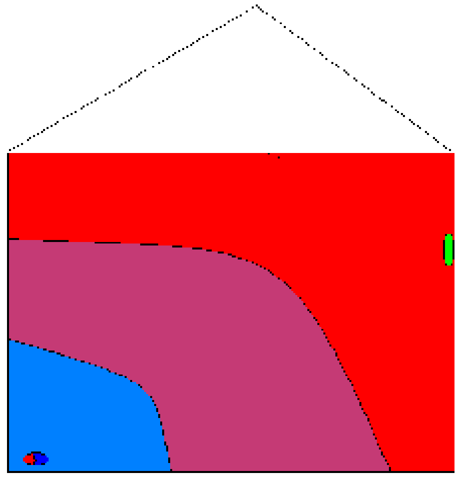
DELAY DIFFERENTIAL EQUATIONS

In a Delay Differential Equation, you are working problems in which the "instantaneous rate of change" $x'(t)$ does not depend on the quantity $x(t)$ at that very instant, but really at some other (earlier) time, $x(t-5)$, or later time, $x(t+5)$. In general, functional differential equations express $x'(t)$ in terms of other functions of time, such as the reciprocal, $x(\frac{1}{t})$, or negative time, $x(-t)$, etc.

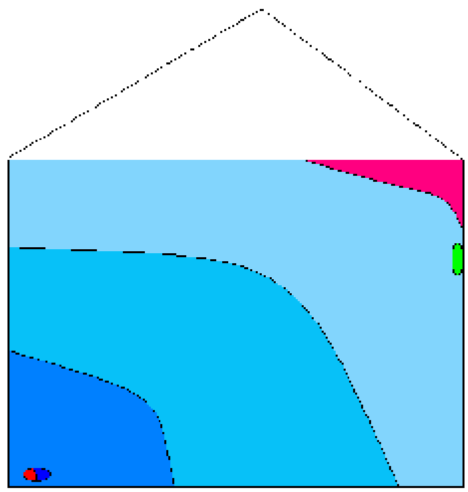
EXAMPLE: THE THERMOSTAT PROBLEM



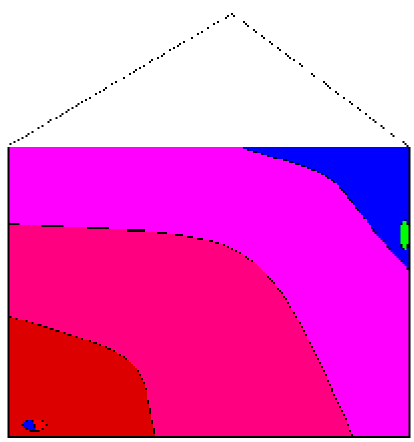
Initial state. Set T to a cooler state.



Room cooling Down



Thermostat now reads a cold room



Room warming up

Suppose some quantity (such as room temperature) depends upon room temperature as controlled by a thermostat.

$$x'(t) = -.16\,x(t-10), t \geq 0 \qquad (5)$$

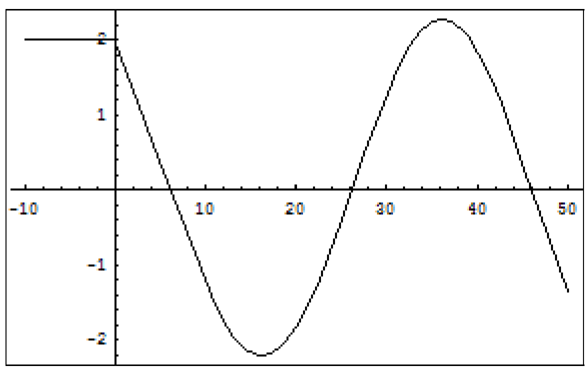
Which says "Read the temperature ten minutes ago and change the temperature by -0.16. This equation looks a little like equation (4), but the solution will be quite different

We really need more information to solve this equation. For Example if we are at time $t=0$, we need to know what the temperature was ten minutes ago, at time $t=-10$, and at all the times from -10 to 0. That is we need to know $x(t)$ for all $t \in [-10, 0]$.

The following graph is the solution to Equation (5), when we are given that

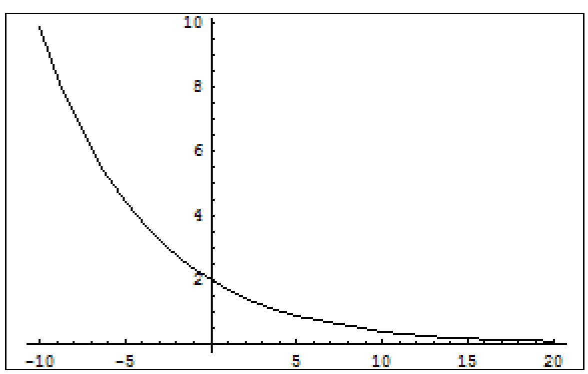
$$x(t) = 2$$

when $-10 \leq t \leq 0$



Solution to $x'(t) = -.16\,x(t-10)$

Compare this to the graph:



Solution to $x'(t) = -.16x(t)$

Methods For Solving Constant Coefficient Delay Equation

If $d > 0$, and $p(t)$ is a known function whose derivatives are continuous on the interval $[-d, 0]$, (that is, $p(t) \in \mathcal{C}^{\infty}[-d, 0]$) and if k is a constant, then the delay differential equation

$$x'(t) = k\,x(t-d), t > 0 \quad (6)$$

$$x(t) = p(t), t \in [-d, 0] \quad (7)$$

can be solved by
1. The method of Characteristics
2. LaPlace transforms
3. Method of Steps

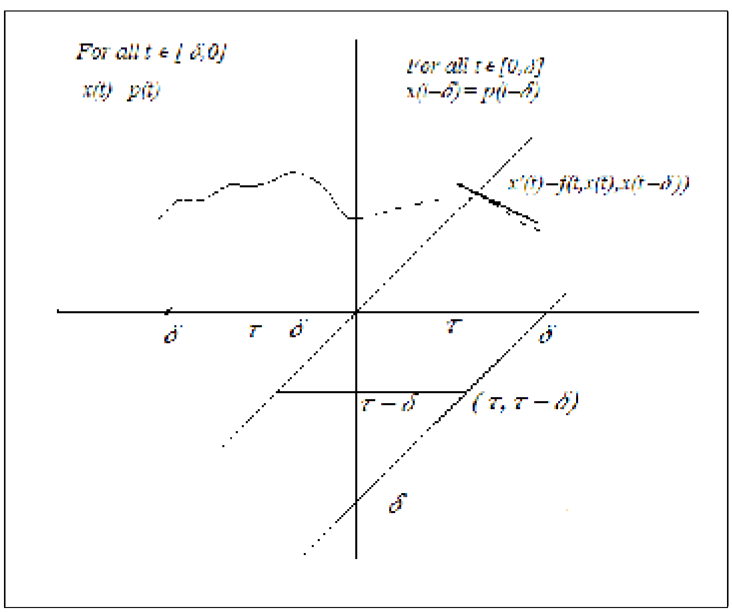
THE METHOD OF STEPS

Skipping the first two methods, we discuss the method of steps; it is the easiest and it works for more general equations.
Let $\delta > 0$, and $p(t)$ be a known function $\in \mathcal{C}^1[-\delta, 0]$. The problem is to find a function $x(t)$ for $t \geq 0$ such that

$$x'(t) = f(t, x(t), x(t-\delta)) \text{ on } [0, \delta]$$

and

$$x(t) = p(t) \text{ on } [-\delta, 0]$$



Step 1: If $t \in [-\delta, 0]$, then $x(t) = p(t)$, call this part of the solution $x_0(t)$
Step 2: If $t \in [0, \delta]$, then $x(t-\delta) = x_0(t-\delta)$, so we solve $x'(t) = f(t, x(t), x_0(t-\delta))$, call this part of the solution $x_1(t)$.

__Step 3__: If $t \in [\delta, 2\delta]$, then $x(t-\delta) = x_1(t-\delta)$, so we solve $x'(t) = f(t, x(t), x_1(t-\delta))$, call this part of the solution $x_2(t)$, etc.

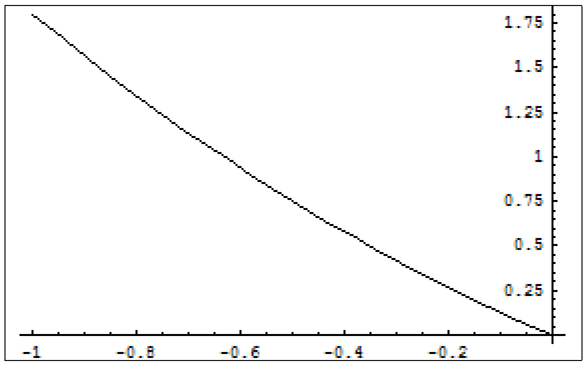
EXAMPLE: If, for $t > 0$,

$$y'(t) = y(t-1) - y(t) \qquad (8)$$

and for $t \in [-1, 0]$

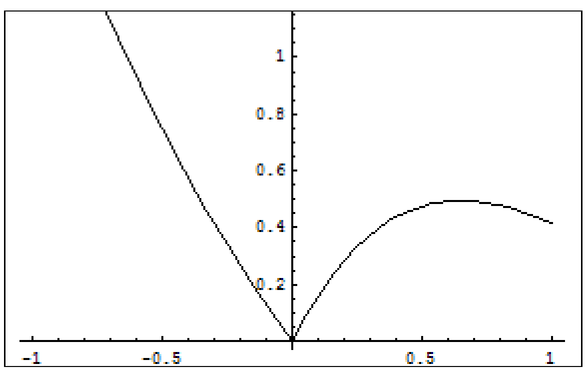
$$y(t) = .6(t-1)^2 - .6 \qquad (9)$$

Which is the step1 solution $y_0(t)$.



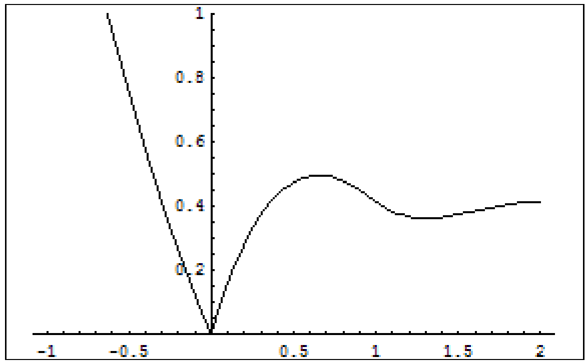
Graph of $y_0(t)$

If $t \in [0,1]$, then $t-1 \in [-1, 0]$, so
$y(t-1) = .6(t-2)^2 - .6$ and the differential equation on $[0,1]$ is:
$y'(t) = .6(t-2)^2 - .6 - y(t)$
Solve for $y(t)$; call the solution $y_1(t)$.



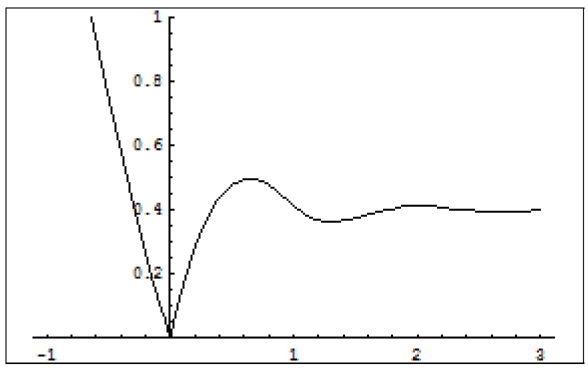
$y_0(t) \cup y_1(t)$

Next for $t \in [1,2]$, $t-1 \in [0,1]$ solve the differential equation $y'(t) = y_1(t-1) - y(t)$



$y_0(t) \cup y_1(t) \cup y_2(t)$

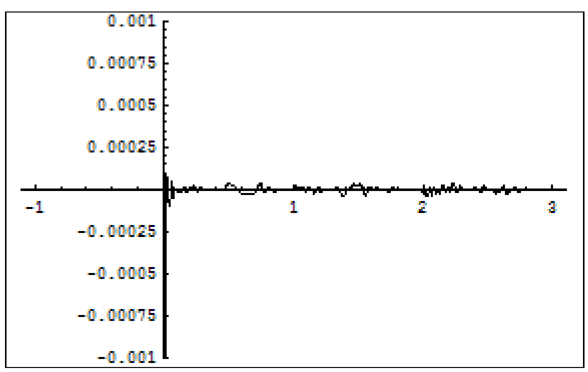
Here is the graphic solution after 3 steps (out to 3).



Solution to $y'(t) = y(t-1) - y(t)$

We check by plotting the graph of

$$y'(t) - y(t-1) + y(t) \text{ on } [-1, 3]$$



Graph of $y'(t) - y(t-1) + y(t)$

Using EXCEL to Solve a Delay Differential Equation

We can use EXCEL to carry out the method of steps, if we, first, change the differential equation

$$x'(t) = f(t, x(t), x(t-\delta), \; t > 0$$

$$x(t) = \phi(t), \; t \leq 0$$

into the finite difference equation:

$$x(t+h) = x(t) + h{*}f(t, x(t), x(t-\delta)), \; t > 0$$

$$x(t) = \phi(t), \; t \leq 0$$

We can then get a graphic solution by using the EXCEL Chart command on Column C as computed in the spread sheet shown in below. Choose $h = \delta/n$, for some integer, say $n = 20$. The delimiter box $\lfloor n \rfloor$notation means use the cell in the nth row. For example $C\lfloor n+1-n{*}h \rfloor$ means use the value in column C and row $n+1-n{*}h$

	A	B	C
1	$= -\delta$	$= \phi(A\lfloor 1 \rfloor)$	$B\lfloor 1 \rfloor$
2	$= A\lfloor 1 \rfloor + h$	$= \phi(A\lfloor 2 \rfloor)$	$B\lfloor 2 \rfloor$
3	$= A\lfloor 2 \rfloor + h$	$= \phi(A\lfloor 3 \rfloor)$	$B\lfloor 3 \rfloor$
...	...	...	...
k	$copy$	$copy$	$copy$
...	...	...	...
$n-1$	$= A\lfloor n-2 \rfloor + h$	$= \phi(A\lfloor n-1 \rfloor)$	$B\lfloor n-1 \rfloor$
n	$= A\lfloor n-1 \rfloor + h$	$= \phi(A\lfloor n \rfloor)$	$B\lfloor n \rfloor$
$n+1$	$= A\lfloor n \rfloor + h$	$= \phi(A\lfloor n+1 \rfloor)$	$C\lfloor n \rfloor + hf(A\lfloor n \rfloor, C\lfloor n \rfloor, C\lfloor n-n{*}h \rfloor)$
$n+2$	$= A\lfloor n+1 \rfloor + h$	$= \phi(A\lfloor n+2 \rfloor)$	$C\lfloor n+1 \rfloor + hf(A\lfloor n+1 \rfloor, C\lfloor n+1 \rfloor, C\lfloor n-(n+1){*}h \rfloor)$
...	...		...

Application:

Pollution Clean-up

Assume a lake has a pollution concentration of $x(t)$ at time t, $t > 0$, where x is the mass/volume. (example: Kg/acre feet).
$V(t)$ is the vol of the lake, usually a constant, say $V = 900$ acre feet.

$p(t) =$ pollution history, mass/vol at time $t < 0$.
$x(t)\,v(t) =$ mass of pollution in the lake at time t. (mass/vol)(vol) $=$ mass.
$(x(t)V(t))' =$ rate of change in pollution, mass/time
With V constant, say v, then $v\,x'(t)$ is the rate of change in pollution, mass/time
r is the constant rate at which the polluted water is removed vol/time.

Assume at time $t = 0$ clean water (0 pollution) is added at a rate of r also. This keeps the volume constant.

Two Models

1. "Instantaneous" or the well-stirred model depicted in biology, physics and engineering articles by a little propeller. The incoming water is instantaneously mixed and the whole lake is uniformly at the new pollution level as it leaves the lake.

$$v\,x'(t) = \,-\,r\,x(t),\ t > 0$$

$$x(0) = x0, \text{ initial pollution.}$$

2. "Delay" model

$$v\,x'(t) = \,-\,r\,x(t-d), t > 0$$

$$x(t) = p(t),\ t < 0$$

$x(t-d) =$ pollution at a time $t-d$ in the past where d is the time it takes for the new mixture to start leaving the lake.

Assume the historical pollution is cut off at time $t = 0$ and clean water is input at a rate of r (vol/time) and suppose $k = r/V$, then the instantaneous equations are:

$$x'(t) = \,-\,k\,x(t),\ t > 0$$

$$x(0) = x_0$$

And the delay equations are:

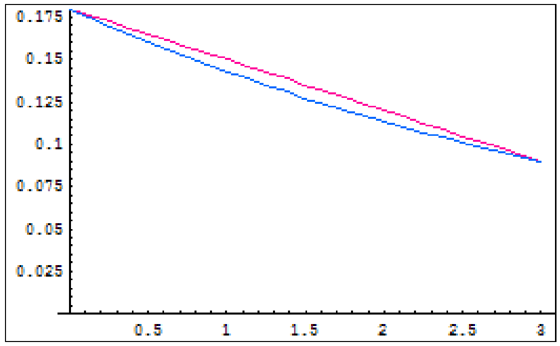
$$x'(t) = \,-\,k\,x(t-d),\ t > 0$$

$$x(t) = p(t),\ t < 0$$

$$p(0) = x_0$$

$p(t)$ is the history of pollution.

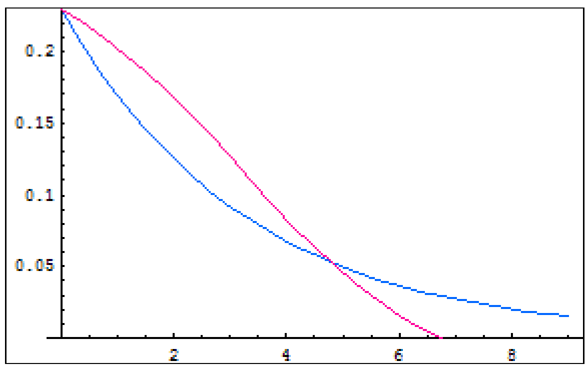
Given a constant history $p(t) = x_0$, say 18% to be reduced to 9% in 3 years, then the solutions to the two models is as follows.



Delay in red

with the well-mixed model requiring $k = .23$, each year 23% of the lakes volume has to be replaced by clear water coming in and polluted going out. For the delay model, $k = .17$, requiring a smaller exchange to achieve the same reduction. Why?

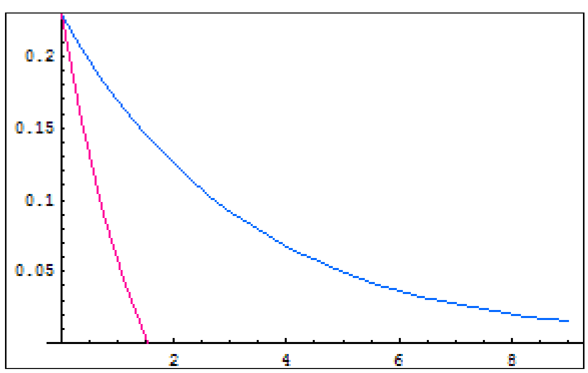
If the pollution history was increasing, say $p(t) = x_0 e^{.2\,t}$, then the older water was cleaner; so the two solutions are:



Delay in red

The well-mixed model assumes that the whole lake has the same level of pollution, but the delay model is getting rid of the older (cleaner) water first so it is slower for a while.

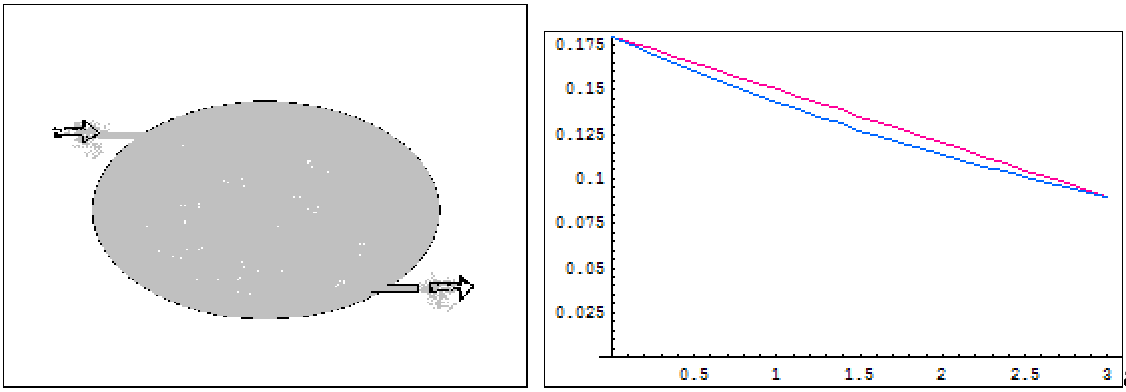
If the pollution history is decreasing, say $p(t) = 0.2\,t^2 + x_0,$ then the solutions are:



Delay in red

The delay model is getting rid of the older dirtier water first, the well-mixed model assumes that all the old water is already mixed and is slower to get rid of it.

Lake Pollution Profiles at the Start of Clean-Up
(Red graph is the delay model)



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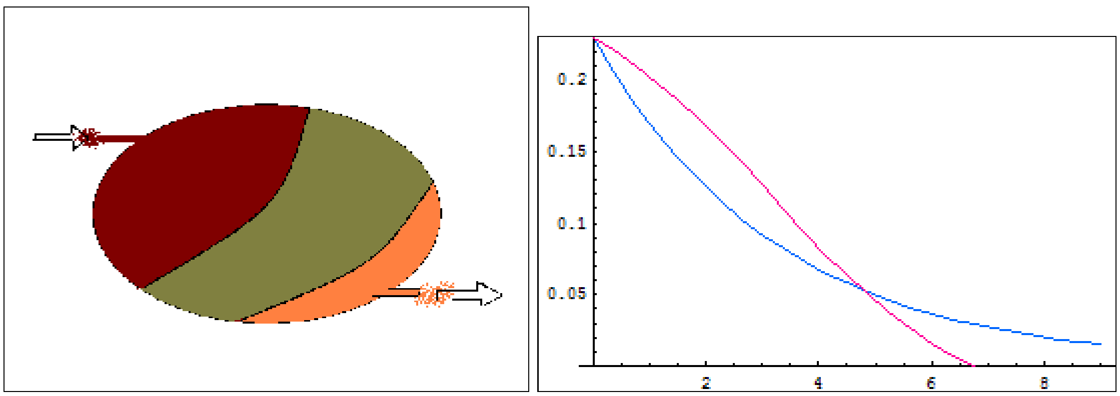
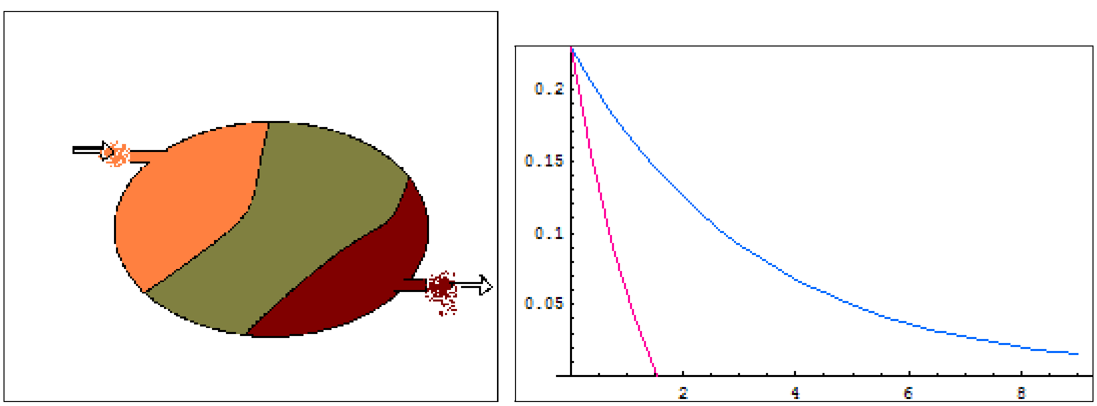
Constant $p(t)$. Different k's to achieve the same target



Increasing $p(t)$, (FIFO of clean water), same k's



Increasing $p(t)$, (FIFO of dirty water), same k's.

HIGHER ORDER DIFFERENTIAL EQUATIONS

A second order equation

$$y''(t) = f(t, y(t), y'(t))$$

could be written as a system of first order equations:

$$y'(t) = u(t)$$
$$u'(t) = f(t, y(t), u(t))$$

Where $\binom{y(t)}{u(t)} = \boldsymbol{v}(t),\ u(t) = g_1(t, y(t), u(t))$, $f(t, y(t), u(t)) = g_2(t, y(t), u(t))$

$$\begin{pmatrix} y(t) \\ u(t) \end{pmatrix}' = \begin{pmatrix} u(t) \\ f(t, y(t), u(t)) \end{pmatrix} = \begin{pmatrix} g_1(t, y(t), u(t)) \\ g_2(t, y(t), u(t)) \end{pmatrix}$$

$$\binom{g_1(t,y(t),u(t))}{g_2(t,y(t),u(t))} = \boldsymbol{g}(t, \boldsymbol{v}(t))$$

$$\boldsymbol{v'}(t) = \boldsymbol{g}(t, \boldsymbol{v}(t))$$

This also holds for higher ordered equations, so equation (1) can represent a very general situation.

Example of a 2nd order <u>delay</u> d.e.

If, for $t > 0$,
$x''(t) = ax(t) + cx'(t) + bx(t-d) + hx'(t-d)$
and for $-d < t < 0$,
$x(t) = p(t),\ x'(t) = q(t)$

Let

$$x'(t) = y(t)$$

then

$$y'(t) = ax(t) + cy(t) + bx(t-d) + hy(t-d)$$

or

$$\binom{x(t)}{y(t)}' = \begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix} \binom{x(t)}{y(t)} + \begin{pmatrix} 0 & 0 \\ b & h \end{pmatrix} \binom{x(t-d)}{y(t-d)}$$

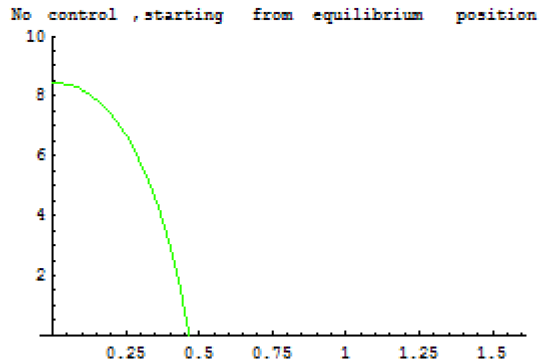
Now if we let $\binom{x(t)}{y(t)} = \boldsymbol{v}(t),$ and
$\boldsymbol{P}(t) = \binom{p(t)}{q(t)}$ then

$v'(t) = A\boldsymbol{v}(t) + B\boldsymbol{v}(t-d)$ for $t > 0$
$\boldsymbol{v}(t) = \boldsymbol{P}(t)$ for $-d < t < 0$

INVERTED PENDULUM -- DELAY CONTROLLERS

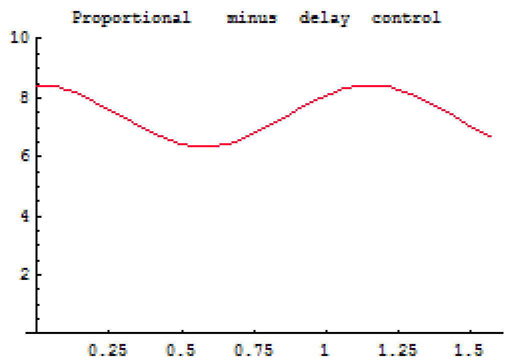
With no controllers

$$x''(t) = ax(t)$$

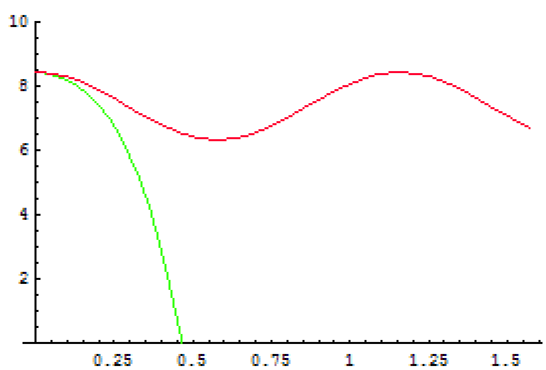


With Proportional minus delay controller (PMD)

$$x''(t) = ax(t) - bx(t-d)$$



Comparison



The vertical axis is the angle of the pendulum (in units of $\pi/16$) and the horizontal axis is time.

Other second order acceleration delay problems are applied to using "shock absorber" connections between buildings to dampen the effect of an earthquake.

$$x''(t) = f(t, x(t), x'(t)) - g(t)x(t-d)$$